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Zero-rest-mass scalar fields for certain space-times

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Abstract : In this paper an exact solution of Einstein—Maxwell-scalar field equations for an axi-symmetric static metric is obtained which includes Melvin's magnetic universe as a special case when scalar field is absent. Further an exact solution of Einstein's field equations for zero-rest-mass scalar field has also been found for a stationary cylindrically symmetric space-time.

1. Introduction

The study of relativistic field equations in the presence of scalar meson field has drawn the attention of many workers and various aspects of the problem have been investigated by Brahmachary (1960), Bergmann and Leipnik (1957), Buchdahl (1959), Janis *et al* (1968), Penney (1968), Gautreau (1969) and others. The existence of the solution of coupled electromagnetic and scalar fields in general relativity is of considerable physical interest and it may serve as a starting point for further investigations.

In Einstein's field equations

$$R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = -8\pi T_\mu^\nu. \tag{1.1}$$

we take the energy-momentum tensor T_μ^ν as

$$T_\mu^\nu = E_\mu^\nu + K_\mu^\nu.$$

Here E_μ^ν and K_μ^ν are the energy-momentum tensors corresponding to the electromagnetic and zero-rest-mass scalar fields respectively.

The tensor E_μ^ν is given by

$$E_\mu^\nu = \frac{1}{4\pi}\left(-F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{4}\delta_\mu^\nu F_{\alpha\beta}F^{\alpha\beta}\right), \tag{1.2}$$

where the electromagnetic field tensor $F_{\mu\nu}$ satisfies Maxwell's equations

$$F^{\mu\nu}_{;\nu} = 0. \tag{1.3}$$

Here a comma and a semicolon stand for an ordinary and a covariant differentiation respectively. The tensor $K_\mu{}^\nu$ is given by

$$K_\mu{}^\nu = \frac{1}{4\pi}\left(V_{;\mu}V^{;\nu} - \frac{1}{2}\delta_\mu{}^\nu V_{;\alpha}V^{;\alpha}\right), \tag{1.4}$$

where the scalar potential V satisfies

$$g^{\mu\nu}V_{,\mu\nu} = 0. \tag{1.5}$$

2. Melvin magnetic universe

Melsin (1964) has given an exact axisymmetric solution of the combined sourceless Einstein Maxwell field equations in static case which represents a parallel bundle of magnetic flux held together by its own gravitational field. It is described by the metric

$$ds^2 = e^{2\psi}(dt^2 - dr^2 - dz^2) - r^2e^{-2\psi}d\phi^2 \tag{2.1}$$

with

$$\psi = \psi(r) = \log\left(1+\frac{Br^2}{4}\right),$$

where B is the magnetic intensity. His solution shows the persistent local energy-stress concentration which may be taken as the defining characteristic of a geon. Prasanna and Kumar (1973) studied the spin precission of a charged particle in Melvin's magnetic universe including the effects of general relativity.

In this section we have found a solution of the Einstein-Maxwell field equations corresponding to a parallel bundle of magnetic flux coupled with zero-rest-mass scalar field for the axisymmetric static metric (2.1). The co-ordinates r, ϕ, Z, t are levelled as x^1, x^2, x^3, x^4 respectively. Here the electromagnetic field is associated with an arbitrary constant and the scalar field is associated with another arbitrary constant. When the scalar field is absent we get the solution of Melvin (1964). In the absence of magnetic field a solution of coupled Einstein-scalar field equations is obtained.

Now the equations (1.2) and (1.4) give the electromagnetic and scalar energy momentum tensors as

$$8\pi E_{\mu}{}^{\nu} = B^2 e^{-4\psi} \quad \mathrm{diag}(+1,\ -1,\ +1,\ +1) \tag{2.3}$$

$$8\pi K_{\mu}{}^{\nu} = V_1{}^2 e^{-2\psi} \quad \mathrm{diag}(-1,\ +1,\ +1,\ +1) \tag{2.4}$$

where the scalar potential V by symmetry is a function of r alone.

From equations (2.1), (1.1), (2.3) and (2.4), the independent field equations are

$$2\psi_{11} + \psi_1{}^2 = B^2 e^{-2\psi} - V_1{}^2. \tag{2.5}$$

$$\frac{2\psi_1}{r} - \psi_1{}^2 = B^2 e^{-2\psi} + V_1{}^2. \tag{2.6}$$

Equation (1.5) gives

$$V_{11} + \frac{V_1}{} = 0. \tag{2.7}$$

The subscript denotes differentiation with respect to r throughout.

Equations (2.9) and (2.10) are found satisfied by the value of ψ in (2.12) equation provided that

$$4c_1c_2(1-a^2) = B^2. \tag{2.13}$$

The relation (2.13) will determine a, the constant associated with the scalar field. Hence the solution of Einstein-Maxwell-Scalar field equations for the metric (2.1) is

$$ds^2 = \left\{c_1r^{1-\sqrt{(1-a^2)}}+c_2r^{1+\sqrt{(1-a^2)}}\right\}^2(dt^2-dr^2-d_z^2)$$
$$-r^2\left\{c_1r^{1-\sqrt{(1-a^2)}}+c_2r^{1+\sqrt{(1-a^2)}}\right\}^{-2} d\phi^2. \tag{2.14}$$

This solution has a singularity along $r = 0$, which can be identified with the presence of the source. The scalar density increases as r increases. The space-time (2.14) represents a distribution of zero-rest-mass scalar particles in Melvin's magnetic universe.

Case (*i*). When $a = 0$, the scalar field disappears and the solution (2.14) reduces to that of Melvin's magnetic universe.

Case (*ii*). When $B = 0$, the magnetic field is absent and the relation (2.13) gives $a^2 = 1$ if c_1 and c_2 are non-zero. In this case equation (2.12) reduces to

$$e^{\psi} = cr \quad \text{where } c = c_1+c_2 \tag{2.15}$$

and the line element (2.14) takes the form

$$ds^2 = r^2(dt^2-dr^2-d_z^2)-d\phi^2 \tag{2.16}$$

The space-time (2.16), which is not-flat, is the solution of combined Einstein-Scalar field equations for the metric (2.1). If however c_1 or c_2 be zero, we get a solution for Einstein-Scalar field for arbitrary value of $a^2 < 1$.

4. Stationary cylindrically symmetric space-time

In this section an exact solution of Einstein's field equations for coupled gravitational and zero-rest-mass scalar fields has been found for the stationary cylindrically symmetric space-time (Som and Raychaudhury 1968)

$$ds^2 = dt^2-e^{2\psi}(dr^2+d_z^2)-ld\phi^2+2md\phi dt, \tag{3.1}$$

where ψ, l and m are functions of r alone. For this stationary metric the scalar field V, by symmetry, is a function of r only.

From (1.4) and (3.1), it follows that

$$T_1^1+T_2^2 = 0 \tag{3.2}$$

and hence from (1.1)

$$R_3^3+R_4^4 = 0. \tag{3.3}$$

We can now introduce Weyl—like canonical coordinate system in this stationary case such that (Van Stockum 1937)

$$l+m^2 = r^2. \tag{3.4}$$

For the metric (3.1), in view of (1.1), (1.4) and (3.4), we can write the field equations explicitly as

$$\frac{\psi_1}{r}+\left(\frac{m_1}{2r}\right)_2 = \frac{1}{2}KV_1^2. \tag{3.5}$$

$$-2mm_{11}-\frac{m_1^2}{2}-\frac{3}{2}\left(\frac{mm_1}{r}\right)^2+\frac{2mm_1}{r}+2(m^2-r^2)\psi_{11} = K(r^2-m^2)V_1^2. \tag{3.6}$$

$$-\frac{3}{2}\left(\frac{m_1}{r}\right)^2+2\psi_{11} = -KV_1^2. \tag{3.7}$$

$$-m_{11}+\frac{m_1}{r}-\frac{3}{2}m\left(\frac{m_1}{r}\right)^2+2m\psi_{11} = -KmV_1^2. \tag{3.8}$$

From (1.5), we get

$$V_{11}+\frac{V_1}{r} = 0. \tag{3.9}$$

Here the subscripts 1 and 11 after ψ, m and V denote first and second derivative respectively with respect to r throughout.

The general solution of the equation (3.9) is given by

$$V = \alpha\log r+\beta, \tag{3.10}$$

where α and β are arbitrary constants associated with the scalar field V.
From equations (3.7) and (3.8), we get

$$m_{11}-\frac{m_1}{r} = 0. \tag{3.11}$$

Equations (3.6) and (3.7) give

$$m\left(m_{11}-\frac{m_1}{r}\right)+m_1^2 = 0. \tag{3.12}$$

Combining equations (3.11) and (3.12) and integrating, we obtain $m_1 = 0$, which implies that

$$m = a, \tag{3.13}$$

where a is an arbitrary constant.

Substituting for V and m in equation (3.5) and integrating, we get

$$C^{2\psi} = (\delta . r)^b, \tag{3.14}$$

δ being an arbitrary constant and $b = k\alpha^2$ (constant). Now equation (3.4) gives

$$l = (r^2 - a^2). \tag{3.15}$$

Thus the solution of Einstein's field equations for zero-rest-mass scalar field corresponding to the stationary cylindrically symmetric metric (3.1) is

$$ds^2 = dt^2 - r^b(dr^2 + dz^2) - (r^2 - a^2)d\phi^2 + 2ad\phi dt \tag{3.16}$$

where a and b are constants. Now the stationary solution (3.16), due to (3.13) and the substitution $\bar{t} = t + a\phi$, reduces to the static form

$$ds^2 = d\bar{t}^2 - r^b(dr^2 + dz^2) - r^2 d\phi^2. \tag{3.17}$$

From equations (3.10), (3.14) and (3.15), it is obvious that the solution obtained gives divergent gravitational and massless scalar fields; and as $r \to \infty$, V, ψ and l all the three tend to ∞. It may be noted that the above solution has a singularity along $r = 0$, which can be identified with the presence of the source. Also along $r = 0$, we get a two dimensional flat hypersurface on which the gravitational and zero-rest-mass scalar fields both become singular.

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